Asymptotic Behavior of the Scattering Amplitude and Normal and Abnormal Solutions of the Bethe-Salpeter Equation*

NOBORU NAKANISHI
Brookhaven National Laboratory, Upton, New York
(Received 8 May 1964)

An integral equation for a scattering amplitude is considered in the ladder approximation. It is assumed that two scalar particles having mass m exchange scalar photons except for the first step, in which a scalar meson having mass $2m$ is exchanged. The exact solution to this equation is found in a compact form in the case of zero energy. Asymptotic behavior of the solution is investigated in the crossed channel. It is shown that the leading term and the second leading term in the asymptotic expansion in t exactly correspond to the normal solutions of the Bethe-Salpeter equation with $n=l+1$ and those with $n=l+2$, respectively, where n is the principal quantum number.

1. INTRODUCTION

SOME years ago, Regge[1] established, in the nonrelativistic potential theory, that the asymptotic behavior of the scattering amplitude in the crossed channel is closely related to the bound-state solutions of the same Schrödinger equation. Field-theoretical extensions of his analysis have been made in two directions; one is the S-matrix theory based on the assumption of the Mandelstam representation,[2] and the other is the off-the-mass-shell theory based on the Bethe-Salpeter formalism.[3–9] In the latter approach, Bertocchi, Fubini, and Tonin[4] have shown, in the framework of the multiperipheral model, that the leading asymptotic behavior of the scattering amplitude is determined by a homogeneous equation, which is equivalent to the partial-wave Bethe-Salpeter equation. The present author[6,7] has shown that the equivalent result can be obtained by using the perturbation-theoretical integral representation, and extended to almost completely general case as far as scalar particles are concerned. It remains unsolved, however, which solutions of the Bethe-Salpeter equation are related to the leading asymptotic behavior of the scattering amplitude, and furthermore it is unknown whether or not similar behaviors are true also for nonleading terms in the asymptotic expansion of the scattering amplitude. These problems are especially interesting because the Bethe-Salpeter equation has abnormal solutions[10] which have no counterparts in the nonrelativistic potential theory. The purpose of the present paper is to investigate these problems by using an exactly solvable example.

We consider the elastic scattering of two scalar particles having mass m in the ladder approximation. Exchanged particles are assumed to be scalar and massless. This model is particularly interesting because we know the complete set of the solutions to the corresponding Bethe-Salpeter equation.[10] But, unfortunately, the integral equation for this model has no solution because of the appearance of infrared divergence.[6,11] Hence we shall replace the massless particle by a massive meson only in the inhomogeneous term of the integral equation. Then we can exactly solve the integral equation in terms of the perturbation-theoretical integral representation. In the next section, the exact solution is presented in a compact form in case of zero energy ($s=0$). The asymptotic behavior of the solution in the t channel is investigated in Sec. 3. It is shown there that the leading term and the second leading term in the asymptotic expansion in t precisely correspond to the normal solutions of the Bethe-Salpeter equation with $n=l+1$ and those with $n=l+2$, respectively, but the third term does not correspond to those with $n=l+3$. Some remarks are made in Sec. 4.

2. EXACT SOLUTION

We consider the following integral equation:

$$(m^2-v)(m^2-w)f(v,w,t) = \frac{1}{\mu^2-t-i\epsilon}+\frac{\lambda}{\pi^2 i}\int d^4p'\frac{f(v',w',t')}{-(p-p')^2-i\epsilon}, \quad (2.1)$$

where the notations are the same with those in Ref. 6. The Feynman amplitude $f(v,w,t)$ has the following

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] T. Regge, Nuovo Cimento **14**, 951 (1959); **18**, 947 (1960).
[2] For example, see E. J. Squires, *Complex Angular Momentum and Particle Physics* (W. A. Benjamin, Inc., New York, 1963). It contains further references.
[3] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).
[4] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962); C. Ceolin, F. Duimio, R. Stroffolini, and S. Fubini, *ibid.* **26**, 247 (1962); D. Amati, A. Stanghellini, and S. Fubini, *ibid.* **26**, 896 (1962).
[5] N. Nakanishi, Phys. Rev. **130**, 1230 (1963).
[6] N. Nakanishi, Phys. Rev. **133**, B214 (1964).
[7] N. Nakanishi, Phys. Rev. **133**, B1224 (1964).
[8] C. Cosenza, L. Sertorio, and M. Toller, Nuovo Cimento **31**, 1086 (1964), and work to be published.
[9] J. D. Bjorken, J. Math. Phys. **5**, 192 (1964); M. Baker and I. J. Muzinich, Phys. Rev. **132**, 2291 (1963).
[10] G. C. Wick, Phys. Rev. **96**, 1124 (1954); R. E. Cutkosky, *ibid.* **96**, 1135 (1954).
[11] K. Nishijima, Progr. Theoret. Phys. (Kyoto) **14**, 203 (1955).

integral representation:

$$f(v,w,t)=\int_0^1 (1-y)dy\int_{-1}^1 dz\frac{\varphi(y,z)}{\{(1-y)[\frac{1}{2}(1+z)(m^2-v)+\frac{1}{2}(1-z)(m^2-w)]+y(\mu^2-t)-i\epsilon\}^3}. \quad (2.2)$$

It is convenient to introduce a variable x through[6]

$$y=(1+x)^{-1}. \quad (2.3)$$

Then the weight function $\varphi(y,z)\equiv\psi(x,z)$ satisfies the following integral equation:

$$\psi(x,z)=1+\tfrac{1}{2}\lambda\int_{-1}^{1}dz'\times\int_0^{xR(z,z')}dx'\frac{x'\psi(x',z')}{(1+x')\mu^2+x'^2\rho(z')}, \quad (2.4)$$

where

$$R(z,z')\equiv(1\mp z)/(1\mp z') \quad \text{for} \quad z\gtrless z', \quad (2.5)$$

$$\rho(z)\equiv m^2-\tfrac{1}{4}(1-z^2)s. \quad (2.6)$$

The integral equation (2.4) can be solved most easily by expanding $\psi(x,z)$ in powers of x:

$$\psi(x,z)=\sum_{n=0}^{\infty}a_n(z)x^n. \quad (2.7)$$

The coefficients $a_n(z)$ are determined successively. The first two are evidently given by

$$a_0(z)=1,\quad a_1(z)=0. \quad (2.8)$$

In order to obtain the general form of $a_n(z)$, we shall hereafter confine our consideration to the special case in which

$$s=0,\quad \mu=2m, \quad (2.9)$$

and for simplicity we set $m=1$. Then using the Taylor expansions (2.7) and

$$(2+x)^{-2}=\tfrac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)2^{-n}x^n, \quad (2.10)$$

from (2.4) we obtain the recurrence formula

$$a_{n+2}(z)=\frac{\lambda}{8(n+2)}\int_{-1}^{1}dz'\times[\sum_{r=0}^{n}(-1)^{n-r}(n-r+1)2^{-n+r}a_r(z')][R(z,z')]^{n+2},\quad (n\geq 0). \quad (2.11)$$

By using (2.11), $a_{n+2}(z)$ can be calculated successively; for example,

$$a_2(z)=2^{-4}\lambda(1-z^2),$$
$$a_3(z)=-2^{-5}\lambda(1-z^2), \quad (2.12)$$
$$a_4(z)=2^{-6}\lambda(1-z^2)+2^{-10}\lambda(\lambda-2)(1-z^2)^2.$$

The general form of $a_{n+2}(z)$ is found to be

$$a_{n+2}(z)=\sum_{j=0}^{[n/2]}\frac{(-1)^n(n-j)!\prod_{i=0}^{j}[\lambda-i(i+1)]}{2^{n+2j+4}[(j+1)!]^2j!(n-2j)!}(1-z^2)^{j+1}, \quad (2.13)$$

where $[n/2]$ denotes the largest integer not greater than $n/2$. The proof of (2.13) is given in Appendix A.

Substituting (2.13) together with (2.8) in (2.7), we have

$$\psi(x,z)=1+\sum_{j=0}^{\infty}2^{-2j-2}[(j+1)!]^{-2}\times\prod_{i=0}^{j}[\lambda-i(i+1)]h_j(x)(1-z^2)^{j+1}, \quad (2.14)$$

where

$$h_j(x)\equiv\sum_{n=2j}^{\infty}\frac{(-1)^n}{2^{n+2}}\frac{(n-j)!}{j!(n-2j)!}x^{n+2}=\sum_{n=0}^{\infty}\frac{(n+j)!}{j!n!}\left(-\frac{x}{2}\right)^{n+2j+2}=(\tfrac{1}{2}x)^{2j+2}(1+\tfrac{1}{2}x)^{-j-1}. \quad (2.15)$$

Thus

$$\psi(x,z)=1+\sum_{j=0}^{\infty}\frac{\prod_{i=0}^{j}[\lambda-i(i+1)]}{[(j+1)!]^2}\left[\frac{x^2(1-z^2)}{8(2+x)}\right]^{j+1}=\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{n-1}[i(i+1)-\lambda]}{(n!)^2}\left[-\frac{x^2(1-z^2)}{8(2+x)}\right]^n. \quad (2.16)$$

Since (2.16) is a hypergeometric series, one finds

$$\psi(x,z)=F(-\nu,\nu+1;1;-[x^2(1-z^2)/8(2+x)]), \quad (2.17)$$

with

$$\nu\equiv(\lambda+\tfrac{1}{4})^{1/2}-\tfrac{1}{2}. \quad (2.18)$$

Namely, we obtain

$$\varphi(y,z)=F(-\nu,\nu+1;1;-[(1-y)^2(1-z^2)/8y(1+y)])=P_\nu(1+[(1-y)^2(1-z^2)/4y(1+y)]), \quad (2.19)$$

where P_ν stands for the Legendre function of the first kind.

3. ASYMPTOTIC BEHAVIOR

The behavior of $\varphi(y,z)$ near $y=0$ can easily be obtained by using the well-known formula

$$\begin{aligned}F(a,b;c;u)&\\ =[\Gamma(c)\Gamma(b-a)/\Gamma(b)\Gamma(c-a)]&\\ \times(-u)^{-a}F(a,1-c+a;1-b+a;u^{-1})&\\ +[\Gamma(c)\Gamma(a-b)/\Gamma(a)\Gamma(c-b)]&\\ \times(-u)^{-b}F(b,1-c+b;1-a+b;u^{-1}).&\quad(3.1)\end{aligned}$$

On account of $\nu\geq 0$, we have

$$\begin{aligned}\varphi(y,z)\cong&\frac{\Gamma(2\nu+1)}{[\Gamma(\nu+1)]^2}\left[\frac{(1-y)^2(1-z^2)}{8y(1+y)}\right]^\nu\\ &\times F\left(-\nu,-\nu,-2\nu,-\frac{8y(1+y)}{(1-y)^2(1-z^2)}\right)\\ &=C(1-z^2)^\nu y^{-\nu}(1+b_1y+b_2y^2+\cdots),\end{aligned}\quad(3.2)$$

with

$$\begin{aligned}C&\equiv 2^{-3\nu}\Gamma(2\nu+1)[\Gamma(\nu+1)]^{-2},\\ b_1&\equiv\nu[-3+4(1-z^2)^{-1}],\\ b_2&\equiv\nu[\tfrac{1}{2}(9\nu-1)-12(\nu-1)(1-z^2)^{-1}\\ &\quad+16(\nu-1)^2(2\nu-1)^{-1}(1-z^2)^{-2}].\end{aligned}\quad(3.3)$$

For $\nu>1$, the integration over y in (2.2) should be defined by taking Hadamard's finite part Pf as was done previously.[6]

The asymptotic formula (3.2) indicates the existence of Regge poles at $\nu-1$, $\nu-2$, $\nu-3$, $\cdots$. Our problem is to compare (3.2) with the Cutkosky solutions[10] of the Bethe-Salpeter equation. The zero-energy Cutkosky solutions with $n=l+1$ are

$$g_{\kappa nl}(z)=(1-z^2)^nC_\kappa^{n+\frac{1}{2}}(z),\quad(3.4)$$

with eigenvalues

$$\lambda=(n+\kappa)(n+\kappa+1).\quad(3.5)$$

Here κ, n, l are conventional notations of quantum numbers, and C_κ^m denotes the Gegenbauer polynomial. The solutions with $\kappa=0$ are called normal solutions, and the others ($\kappa=1,2,\cdots$) are called abnormal solutions, which have no counterparts in the nonrelativistic potential theory. It is easy to see that the leading term of (3.2) precisely corresponds to the normal solution with $n=l+1$ when ν is identified with n.

In order to discuss nonleading terms, we must find the asymptotic expansion of $f(v,w,t)$ itself in the t channel. Let

$$I\equiv\int_0^1(1-y)dy\sum_{j=0}^{[\nu]}b_j\,\mathrm{Pf}y^{-\nu+j}/[A+(B+R)y]^3.\quad(3.6)$$

We want to find the asymptotic expansion of I in powers of R. Since

$$\int_1^\infty dy\frac{y^{-k}}{[A+(B+R)y]^3}=O(R^{-3})\quad\text{for}\quad k\geq 0,\quad(3.7)$$

it is possible to replace the upper limit of the integration range in (3.6) by infinity. Then using

$$\begin{aligned}\int_0^\infty dy\frac{\mathrm{Pf}y^{-\nu+j}}{[A+(B+R)y]^3}&=\tfrac{1}{2}\Gamma(-\nu+j+1)\Gamma(\nu-j+2)\\ &\quad\times A^{-\nu+j-2}(B+R)^{\nu-j-1},\end{aligned}\quad(3.8)$$

we easily obtain

$$\begin{aligned}I\cong\tfrac{1}{2}\sum_{k=0}^{[\nu]}\Bigg[\sum_{j=0}^{k}&(b_j-b_{j-1})\Gamma(-\nu+j+1)\Gamma(\nu-j+2)\\ &\times\frac{\Gamma(\nu-j)}{\Gamma(\nu-k)(k-j)!}B^{k-j}A^{-\nu+j-2}\Bigg]R^{\nu-1-k}\end{aligned}\quad(3.9)$$

with $b_{-1}\equiv 0$. Now, the asymptotic expansion of $f(v,w,t)$ in t can be calculated by setting

$$\begin{aligned}A&=1-\tfrac{1}{2}(1+z)v-\tfrac{1}{2}(1-z)w-i\epsilon,\\ B&=-A+4,\\ R&=-t,\end{aligned}\quad(3.10)$$

from (2.2) and (3.2) together with (3.3).

$$\begin{aligned}f(v,w,t)\cong\tfrac{1}{2}\Gamma(-\nu+1)\Gamma(\nu+2)C\int_{-1}^{1}(1-z^2)^\nu dz\Bigg\{&A^{-\nu-2}(-t)^{\nu-1}+4(\nu-1)\left[A^{-\nu-2}+\frac{\nu}{2(\nu+1)}\left(1-\frac{2}{1-z^2}\right)A^{-\nu-1}\right]\\ &\times(-t)^{\nu-2}+8(\nu-1)(\nu-2)\Bigg[A^{-\nu-2}+\frac{\nu}{\nu+1}\left(1-\frac{2}{1-z^2}\right)A^{-\nu-1}\\ &+\frac{1}{\nu+1}\left(\tfrac{1}{4}(\nu+1)-\frac{\nu-1}{1-z^2}+\frac{2(\nu-1)^2}{(2\nu-1)(1-z^2)^2}\right)A^{-\nu}\Bigg](-t)^{\nu-3}+\cdots\Bigg\},\end{aligned}\quad(3.11)$$

where A is given by (3.10).

As is shown in Appendix B, the normal solution of the Bethe-Salpeter equation with $n=l+2$ (apart from a solid harmonic) is explicitly given by

$$\int_{-1}^{1} dz\frac{(1-z^2)^n}{A^{n+2}}+\frac{n}{2(n+1)}\times\int_{-1}^{1} dz\frac{(1-z^2)^n-2(1-z^2)^{n-1}}{A^{n+1}}. \quad (3.12)$$

The correspondence between (3.12) and the coefficient of $(-t)^{\nu-2}$ in (3.11) is exact when ν is again identified with n. It is noteworthy that if we consider the expansion of $f(v,w,t)$ in terms of $P_l(\cos\theta)$ instead of t, where $t=2(1-\cos\theta)$, then the above precise correspondence cannot be obtained.

As for the coefficient of $(-t)^{\nu-3}$ in (3.11), the precise correspondence to the normal solution with $n=l+3$ is realized only in the first two terms. The Cutkosky function $g^2{}_{0,n,n-3}(z)$ is much disturbed by the presence of an abnormal solution $g_{2,n,n-1}(z)$, which is degenerate with the former (see Appendix B). Hence, the coefficient of $A^{-\nu}$ does not reproduce $g^2{}_{0,n,n-3}(z)$. Instead, the last two terms in the coefficient of $A^{-\nu}$ are just proportional to $g_{2,n,n-1}(z)$.

4. REMARKS

In this paper, we have obtained the exact solution to (2.1) in case of (2.9), but our analysis can, in principle, be extended to more general cases. The generalization to the case $\mu\neq 2m$ is interesting because then we can investigate the improper limit $\mu\to 0$. Furthermore, we can get rid of the difficulty of degeneracy in the Bethe-Salpeter equation if the solution in case of $s\neq 0$ is found.

We have considered the case $s=0$. It should strictly be distinguished from the case of vanishing total 4-momentum. In the latter case we have $v=w$ in addition to $s=0$, so that the denominator function in (2.2) becomes independent of z. The elastic forward scattering in the t channel corresponds to this case. From (2.16) we obtain

$$f(v,v,t)=2\int_0^1 (1-y)dy \times\frac{F(-\nu,\nu+1;3/2;-(1-y)^2/8y(1+y))}{[(1-y)(1-v)+y(4-t)-i\epsilon]^3}. \quad (4.1)$$

If we consider the on-the-mass-shell case $v=w=1$, then the scattering amplitude exhibits infrared divergence. In perturbation expansion, the term of order λ^n behaves like $(\ln\delta)^n$, where $\delta=v-1$. The exact solution $(1-v)^2f(v,v,t)$ behaves like $\delta^{-\nu}$ as is seen from (3.11), where ν is given by (2.18).

When ν is a positive integer, the exact solution (2.19) becomes a rational function of y. Hadamard's finite part is not well defined for a negative-integral power of y unless then the coefficient vanishes. Therefore $f(v,w,t)$ is divergent in this case. This is physically reasonable because this infinity corresponds to a bound-state pole at $s=0$ of the Feynman amplitude.

Very recently, Tiktopoulos and Treiman[12] have obtained an upper bound for the leading asymptotic behavior of the scattering amplitude in the ladder approximation. Their bound exactly coincides with our exact result $t^{\nu-1}$ with (2.18) apart from the coefficient.

APPENDIX A: PROOF OF (2.13)

We shall first prove the following integral formula:

$$\int_{-1}^{1} dz'[R(z,z')]^{2n+2+k}(1-z'^2)^n=\tfrac{1}{2}(2n+2+k)\frac{n!k!}{(n+k+1)!}\sum_{j=0}^{[k/2]}\frac{(-1)^j(n+k-j)!}{2^{2j}(n+j+1)!(k-2j)!}(1-z^2)^{n+j+1}, \quad (A1)$$

with $n\geq 0$ and $k\geq 0$. We denote the left-hand side of (A1) by I_{nk} and the right-hand side of it by J_{nk}.

$$\begin{aligned} I_{nk}&=(1-z)^{2n+2+k}\int_{-1}^{z} dz'\frac{(1+z')^n}{(1-z')^{n+2+k}}+(z\to -z)\\ &=(1-z)^{2n+2+k}\int_0^{(1+z)/(1-z)}\frac{du}{2}u^n\left(\frac{1+u}{2}\right)^k+(z\to -z)\\ &=\frac{2n+2+k}{2^{k+1}}\sum_{j=0}^{k}\frac{{}_kC_j(1+z)^{n+j+1}(1-z)^{n+k-j+1}}{(n+j+1)(n+k-j+1)}. \end{aligned} \quad (A2)$$

On the other hand, J_{nk} can be rewritten as

$$J_{nk}=\frac{2n+2+k}{2^{k+1}}\frac{n!k!}{(n+k+1)!}\sum_{l=0}^{[k/2]}\frac{(-1)^l(n+k-l)!}{(n+l+1)!(k-2l)!}\sum_{i=0}^{k-2l}\frac{(k-2l)!}{i!(k-2l-i)!}(1+z)^{n+l+i+1}(1-z)^{n+k-l-i+1} \quad (A3)$$

[12] G. Tiktopoulos and S. B. Treiman, Phys. Rev. 134, B844 (1964).

by using $1=2^{-k+2l}[(1+z)+(1-z)]^{k-2l}$. Since both expressions (A2) and (A3) are symmetric with respect to $(1+z)$ and $(1-z)$, it is sufficient to compare the coefficients of

$$(1+z)^{n+j+1}(1-z)^{n+k-j+1} \tag{A4}$$

for $j\leq[k/2]$. Since the coefficient of (A4) in (A3) is

$$\frac{2n+2+k}{2^{k+1}}\frac{n!k!}{(n+k+1)!}\sum_{l=0}^{j}\frac{(-1)^l(n+k-l)!}{(n+l+1)!(j-l)!(k-j-l)!}, \tag{A5}$$

we have only to prove the identity

$$\sum_{l=0}^{j}(-1)^l\frac{n!(n+k-l)!j!(k-j)!}{(n+l+1)!(n+k+1)!(j-l)!(k-j-l)!}=\frac{1}{(n+j+1)(n+k-j+1)}. \tag{A6}$$

The left-hand side of (A6), which we denote by K, can be written as

$$K=\frac{n!}{n+k+1}\sum_{l=0}^{\infty}\frac{a(a+1)\cdots(a+l-1)\times b(b+1)\cdots(b+l-1)}{c(c+1)\cdots(c+l-1)\times(n+l+1)!}, \tag{A7}$$

with

$$a=-j,\quad b=-k+j,\quad c=-n-k. \tag{A8}$$

Thus

$$\begin{aligned}K&=\frac{n!}{n+k+1}\int_0^1 dz_n\int_0^{z_n}dz_{n-1}\cdots\int_0^{z_1}dz_0F(a,b;c;z_0)\\&=(n+k+1)^{-1}\int_0^1 dz(1-z)^nF(a,b;c;z).\end{aligned} \tag{A9}$$

Since F satisfies the hypergeometric differential equation,

$$z(1-z)(d^2F/dz^2)+[c-(a+b+1)z](dF/dz)-abF=0, \tag{A10}$$

we have

$$\begin{aligned}0&=\int_0^1 dz\left\{z(1-z)^{n+1}\frac{d^2F}{dz^2}+[c-(a+b+1)z](1-z)^n\frac{dF}{dz}-ab(1-z)^nF\right\}\\&=\int_0^1 dz\{-(1-z)^{n+1}+[c-(a+b-n)z](1-z)^n\}\frac{dF}{dz}-ab\int_0^1 dz(1-z)^nF.\end{aligned} \tag{A11}$$

Since (A8) leads to $c=a+b-n$, (A11) yields

$$0=(a+b-n-1)\left[-1+(n+1)\int_0^1 dz(1-z)^nF\right]-ab\int_0^1 dz(1-z)^nF. \tag{A12}$$

Thus

$$\int_0^1 dz(1-z)^nF=\frac{n+1-a-b}{(n+1-a)(n+1-b)}. \tag{A13}$$

Substituting (A13) together with (A8) in (A9), we obtain the right-hand side of (A6). Thus (A1) has been established.

Now, we shall prove (2.13) by mathematical induction starting from (2.8). The recurrence formula (2.11) with (2.8) leads to

$$a_{n+2}(z)=\frac{\lambda}{8(n+2)}\int_{-1}^1 dz'\left[\frac{(-1)^n(n+1)}{2^n}+\sum_{m=0}^{n-2}(-1)^{n-m}\frac{n-m-1}{2^{n-m-2}}a_{m+2}(z')\right][R(z,z')]^{n+2}. \tag{A14}$$

Substitution of (2.13) for a_2, a_3, $\cdots$, a_n in (A14) yields

$$a_{n+2}(z)=\frac{(-1)^n\lambda}{2^{n+3}(n+2)}\int_{-1}^1 dz'\{(n+1)+\sum_{j=0}^{[n/2]-1}(A_{nj}\prod_{i=0}^{j}[\lambda-i(i+1)]/2^{2j+2}[(j+1)!]^2)(1-z'^2)^{j+1}\}[R(z,z')]^{n+2}, \tag{A15}$$

with

$$A_{nj}\equiv\sum_{m=2j}^{n-2}\frac{(m-j)!}{j!(m-2j)!}(n-m-1)$$
$$=\frac{1}{j!}\left[(n-2j-1)\sum_{r=0}^{n-2j-2}\frac{(r+j)!}{r!}-\sum_{r=1}^{n-2j-2}\frac{(r+j)!}{(r-1)!}\right].\quad\text{(A16)}$$

We make use of the well-known formula

$$\sum_{i=1}^{n}\frac{(i+m-1)!}{(i-1)!}=\frac{1}{m+1}\frac{(n+m)!}{(n-1)!},\quad\text{(A17)}$$

which can easily be verified by mathematical induction. Then

$$A_{nj}=\frac{1}{j!}\left[\frac{(n-2j-1)\times(n-j-1)!}{(j+1)\times(n-2j-2)!}-\frac{(n-j-1)!}{(j+2)\times(n-2j-3)!}\right]$$
$$=\frac{(n-j)!}{(j+2)!(n-2j-2)!}.\quad\text{(A18)}$$

Thus

$$a_{n+2}(z)=\frac{(-1)^n\lambda}{2^{n+3}(n+2)}\int_{-1}^{1}dz'\{\sum_{r=0}^{[n/2]}((n-r+1)!\prod_{i=0}^{r-1}[\lambda-i(i+1)]/2^{2r}(r!)^2(r+1)!(n-2r)!)(1-z'^2)^r\}[R(z,z')]^{n+2}.\quad\text{(A19)}$$

Now, we apply the integral formula (A1) to (A19), and get

$$a_{n+2}(z)=\frac{(-1)^n\lambda}{2^{n+4}}\sum_{j=0}^{[n/2]}\frac{(-1)^j(n-j)!B_j}{2^{2j}(j+1)!(n-2j)!}(1-z^2)^{j+1},\quad\text{(A20)}$$

with

$$B_j\equiv\sum_{r=0}^{j}\frac{(-1)^r}{r!(r+1)!}\prod_{i=0}^{r-1}[\lambda-i(i+1)].\quad\text{(A21)}$$

By mathematical induction, we can easily prove

$$B_j=\frac{(-1)^j}{j!(j+1)!}\prod_{i=1}^{j}[\lambda-i(i+1)].\quad\text{(A22)}$$

Substitution of (A22) in (A20) yields

$$a_{n+2}(z)=(-1)^n\sum_{j=0}^{[n/2]}((n-j)!\prod_{i=0}^{j}[\lambda-i(i+1)]/2^{n+2j+4}[(j+1)!]^2j!(n-2j)!)(1-z^2)^{j+1}.\quad\text{(A23)}$$

Thus (2.13) has been proved.

APPENDIX B: ZERO-ENERGY CUTKOSKY SOLUTIONS

The Cutkosky solutions are represented as[10]

$$\sum_{k=0}^{n-l-1}\int_{-1}^{1}dz\frac{g^k{}_{\kappa nl}(z)}{[1-\frac{1}{2}(1+z)v-\frac{1}{2}(1-z)w-i\epsilon]^{n-k+2}}\quad\text{(B1)}$$

apart from a solid harmonic. In case of zero energy, $g^0{}_{\kappa nl}=g_{\kappa nl}$ is given by (3.4) for any l. The eigenvalues of λ are given by (3.5), which is independent of l. The other weight functions $g^k{}_{\kappa nl}$, $(k>0)$, satisfy inhomogeneous integral equations:

$$g^k{}_{\kappa nl}(z)=\frac{\lambda}{2(n-k)}\sum_{j=0}^{k}\frac{(n-k+1)!(n-j-l-1)!}{(n-j+1)!(n-k-l-1)!}\int_{-1}^{1}dz'[R(z,z')]^{n-k}g^j{}_{\kappa nl}(z'),\quad\text{(B2)}$$

where $R(z,z')$ is defined by (2.5).

Thus $g^1{}_{0nl}$ is determined by

$$g^1{}_{0nl}(z)=\frac{n(n-l-1)}{2(n-1)}\int_{-1}^{1}dz'[R(z,z')]^{n-1}(1-z'^2)^n+\frac{n(n+1)}{2(n-1)}\int_{-1}^{1}dz'[R(z,z')]^{n-1}g^1{}_{0nl}(z'). \tag{B3}$$

The homogeneous part of (B3) is satisfied by $g^0{}_{1,n-1,l}$, but it is an odd function of z so that it is quite harmless. By making an ansatz

$$g^1{}_{0nl}(z)=c_0(1-z^2)^n+c_1(1-z^2)^{n-1}, \tag{B4}$$

we can easily obtain

$$g^1{}_{0nl}(z)=\frac{n(n-l-1)}{2(n+1)}[(1-z^2)^n-2(1-z^2)^{n-1}]. \tag{B5}$$

As for $g^2{}_{0nl}$, the homogeneous part of the integral equation for it is satisfied by $g^0{}_{2,n-2,l}$, which is an *even* function of z. Hence, we can no longer find a solution of the type

$$c_0(1-z^2)^n+c_1(1-z^2)^{n-1}+c_2(1-z^2)^{n-2} \tag{B6}$$

as is easily checked.

PHYSICAL REVIEW VOLUME 135, NUMBER 6B 21 SEPTEMBER 1964

Unitary Symmetry and Hyperon Leptonic Decays*

L. Wolfenstein
Carnegie Institute of Technology, Pittsburgh, Pennsylvania
(Received 13 May 1964)

The consequences (beyond the $\Delta I=\frac{1}{2}$ and $\Delta I=1$ rules) of the most general form of the hypothesis that the weak-interaction currents transform like components of SU_3 octets are discussed. These are presented as relations between the $\Delta S=1$ leptonic decays of Ξ^- and Ξ^0 and those of Σ and Λ. The only prediction for the branching ratios B of Ξ decays which can be compared to present experiments is

$$B(\Xi^-\to\Lambda+e^-+\nu)+B(\Xi^-\to\Sigma^0+e^-+\nu)\leq(1.05\pm0.2)\times10^{-3}.$$

As yet the comparison is inconclusive. One additional relation among the leptonic decays of hyperons is found if the particular model of Cabibbo is assumed. Applications of these considerations to the determination of induced couplings are made.

1. INTRODUCTION

THE success of the unitary symmetry model for strong interactions has led many authors to suggest possible properties of the weak-interaction currents with respect to the SU_3 transformations.[1–4] Practically all the proposals include the hypothesis that each of the nonleptonic weak currents which are coupled to leptons (or to intermediate bosons) transform like components of some octet. The main purpose of the present note is to discuss those experimentally observable consequences that follow from this hypothesis alone and so are common to all the proposals. The present discussion is limited to leptonic decays of hyperons, which are particularly suitable for testing this hypothesis. There exist sixteen possible leptonic decay amplitudes, of which twelve should be observable in the absence of selection rules; the other four either compete with the electromagnetic decay of the Σ^0 or are intrinsically very rare because of their small energy release.

We first review in Sec. 2 those selection rules that may follow from postulating transformation properties of the weak currents with respect to strangeness, isotopic spin, and G. These selection rules, which are well known but not well verified experimentally, provide eight relationships among the sixteen amplitudes. In Sec. 3 we discuss the consequences of the most general form of the octet hypothesis, which provides four additional relationships. The further hypotheses that can be made in an invariant way assuming perfect SU_3 symmetry are discussed in Sec. 5; they lead essentially to the model of Cabibbo, which provides one additional relationship when only hyperon leptonic decays are considered.

The weak interaction responsible for leptonic decays

* This work supported in part by U. S. Atomic Energy Commission.
[1] M. Gell-Mann, California Institute of Technology Report CTSL-20, 1961 (unpublished).
[2] N. Cabibbo and R. Gatto, Nuovo Cimento 21, 872 (1961).
[3] N. Cabibbo, Phys. Rev. Letters 10, 531 (1963); see also B. d'Espagnat and J. Prentki, Nuovo Cimento 24, 497 (1962).
[4] John M. Cornwall and V. Singh, Phys. Rev. Letters 10, 551 (1963).